CHAPTER 10 - INFINITE SERIES AND SEQUENCES

10.1. SEQUENCES

$a_n = n^2 + n \qquad (n = 1, 2, 3, \dots)$

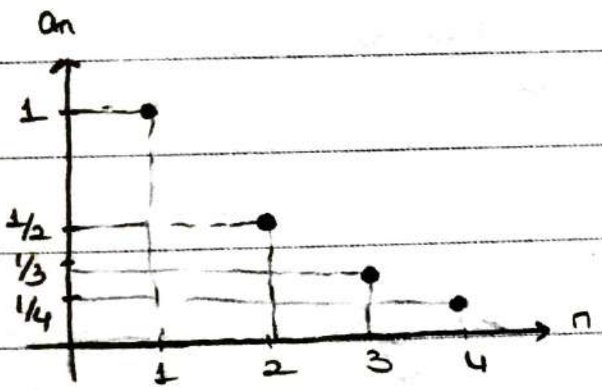
$a_1, a_2, a_3, \dots$

$\downarrow \quad \downarrow \quad \downarrow$

$2 \quad 6 \quad 12 \quad \dots$

* Convergence of sequences

$a_n = \frac{1}{n} \rightarrow 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$



✓ $\lim_{n\to\infty} \frac{1}{n} = 0$ (convergent)

✓ $\lim_{n\to\infty} n = +\infty$ (divergent)

✓ $\lim_{n\to\infty} (-n) = -\infty$ (divergent)

✓ $\lim_{n\to\infty} (-2)^n = ?$ $(-2, 4, -8, 16)$

↳ no limit diverges

✓ $\lim_{n\to\infty} (-1)^n = ?$ $(-1, 1, -1, 1)$

↳ no limit diverges

✓ $\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = ?$ $\left(-\frac{1}{2}, \frac{1}{4}, \frac{-1}{8}\right)$

! $\lim_{n\to\infty} x^n = \begin{cases} +\infty, & x > 1 \\ 1, & x = 1 \\ 0, & -1 < x < 1 \\ \text{diverges}, & x \leq -1 \end{cases}$

from calculus 1

$$\left.\begin{array}{l}\lim\limits_{x\to a} f(x) = L \\ \\ \lim\limits_{x\to a} g(x) = M\end{array}\right\} \lim_{x\to a} \left(f(x) \pm g(x)\right) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$$

$\times$, $\div$ → if $M \neq 0$

$$0 = \lim_{x\to 0} 0 = \lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x^2}\right) = \cancel{\lim_{x\to 0} \frac{1}{x^2} - \lim_{x\to 0} \frac{1}{x^2}}$$

WRONG

calculus 2

$$\left.\begin{array}{l}\lim\limits_{n\to\infty} a_n = L \\ \\ \lim\limits_{n\to\infty} b_n = M\end{array}\right\rangle \text{exist} \Rightarrow \lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$\times$, $\div$ → if $M \neq 0$

ex: $$\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{4/n^6 - 7}{1 + 3/n^6} = \frac{\lim\limits_{n\to\infty} 4/n^6 - 7}{\lim\limits_{n\to\infty} 1 + 3/n^6} = \frac{0-7}{1+0} = -7$$

from calculus 1

$$\left.\begin{array}{l} f(x) \le g(x) \le h(x) \\ \\ \lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L \end{array}\right\} \Rightarrow \lim_{x\to a} g(x) = L$$

sandwich theorem

calculus 2

$a_n \le b_n \le c_n$ for all n after some N

$$\left.\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L\right\} \lim_{n\to\infty} b_n = L$$

ex: $a_n = \dfrac{(-1)^n}{n}$ show $\lim\limits_{n\to\infty} a_n = 0$ by using Sandwich Theorem

Theorem: If f is continuous and $\lim\limits_{n\to\infty} a_n = L$ then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

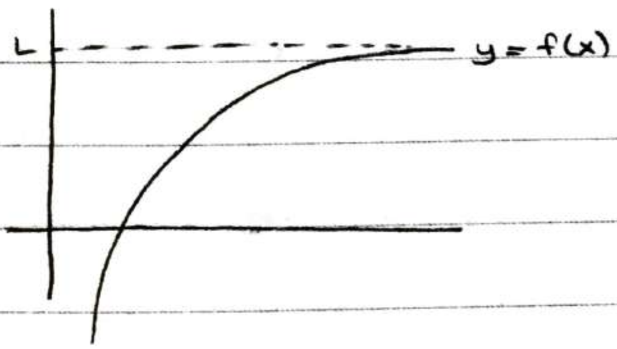
ex: $a_n = \sqrt{\dfrac{n+1}{2n}}$ $\quad \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt{\dfrac{n+1}{2n}}$

$$= \sqrt{\lim_{n\to\infty} \frac{n+1}{2n}} = \sqrt{\frac{1}{2}}$$

since $\sqrt{x}$ is continuous for $x \geq 0$

ex: $\lim\limits_{n\to\infty} 2^{1/n} = 2^{\lim\limits_{n\to\infty} 1/n} = 2^0 = 1$

since $x \to 2^x$ is continuous



$\lim\limits_{x\to\infty} f(x) = L$ and $a_n = f(n)$

$\Rightarrow \lim\limits_{n\to\infty} a_n = L$

ex: $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = ?$ $\quad \left(\dfrac{\ln 1}{1}, \dfrac{\ln 2}{2}, \dfrac{\ln 3}{3} \ldots\right)$

$\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$ $\left(\dfrac{\infty}{\infty}\right)$

$= \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0 \longrightarrow \lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

** ex: $a_n = \left(\frac{n+1}{n-1}\right)^n \quad n \geq 2$

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n$$

$$= n \ln\left(\frac{n+1}{n-1}\right)$$

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\left(\frac{n+1}{n-1}\right)} \cdot \frac{[(n-1)-(n+1)\cdot 1]}{(n-1)^2}}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{-2}{(n+1)(n-1)}}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2$$

$$= \lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty} \ln a_n} = e^2$$

** ex: $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

1 year %100	6M %50
$S(1+1) = 2S$	$S(1+\frac{1}{2})(1+\frac{1}{2}) = 2.25S$
4M %100/3	
$S(1+\frac{1}{3})(1+\frac{1}{3})(1+\frac{1}{3}) \to S(1+\frac{1}{3})^3$	

$$= \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n}$$

$$= e^{\lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)}$$

$$= e^{\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/n}} \quad \left[\frac{0}{0}\right]$$

$$= e^{\lim_{n\to\infty} \left[\frac{\left(\frac{1}{1+1/n}\right)\left(-1/n^2\right)}{(-1/n^2)}\right]} \Rightarrow e^{\lim_{n\to\infty} \frac{1}{1+1/n}} = e^1 = 2.71$$

Some Important Limits

- $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

- $\lim\limits_{n\to\infty} x^{1/n} = x^0 = 1$

- $\lim\limits_{n\to\infty} n^{1/n} = 1 \longrightarrow \lim\limits_{n\to\infty} n^{k/n} = 1$

- $\lim\limits_{n\to\infty} x^n = \begin{cases} +\infty & , \quad x > 1 \\ 1 & , \quad x = 1 \\ 0 & , \quad |x| < 1 \\ \text{diverges} & , \quad x \leq -1 \end{cases}$

- $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$, for all x

- $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ for all x

Recursive Sequences

ex: $a_n = n \cdot a_{n-1}$, $n \geq 1$

$$\left.\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_1 = 2.1 \\ a_3 &= 3.a_2 = 3.2.1 \\ a_4 &= 4.a_3 = 4.3.2.1 \end{aligned}\right\} \quad \underline{\underline{a_n = n!}}$$

ex: Fibonacci

$a_1 = 1 \qquad a_2 = 1$

$a_{n+1} = a_n + a_{n-1} \quad , \quad n \geq 2$

$a_3 = 2$

$a_4 = 3$

$a_5 = 5$

$a_6 = 8$

$a_7 = 13$

- if $a_{n+1} \geq a_n$, for all n

  a_n is nondecreasing.

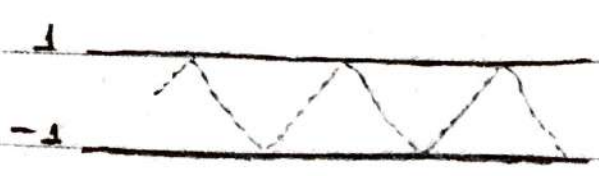
- if there is there is M such that

  $a_n \leq M$ then a_n is called bonded from above sequence

ex: $a_n = \cos(n) \Rightarrow a_n \leq 1$ for all n

a_n is bounded from above

$a_n = \cos(n)$



not convergent, but bounded from above and below

Theorem If a_n is nondecreasing and bounded from above then a_n is convergent, i.e. $\lim_{n\to\infty} a_n$ exists.
(ldest)

(book 32) **ex:** $\lim_{n\to\infty} \frac{n+3}{n^2+5n+6} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{3}{n^2}}{1+\frac{5}{n}+\frac{6}{n^2}} = \frac{0+0}{1+0+0} = 0$

(book 34) **ex:** $\lim_{n\to\infty} \frac{1+n^3}{70-4n^2} = \lim_{n\to\infty} \frac{-3n^2}{-8n} = \lim_{n\to\infty} \frac{3n}{8} = +\infty$

(book 35) **ex:** $\lim_{n\to\infty} 1+(-1)^n$ $(0,2,0,2,0,2 \dots)$

↳ No limit

(book 39) **ex:** $\lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1}$

$$\underbrace{\frac{-1}{2n-1}}_{\lim_{n\to\infty} \frac{-1}{2n-1} = 0} \leq \frac{(-1)^{n+1}}{2n-1} \leq \underbrace{\frac{1}{2n-1}}_{\lim_{n\to\infty} \frac{1}{2n-1} = 0}$$

By sandwich theorem $\Rightarrow \lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1}$

(book 42) **ex:** $\lim_{n\to\infty} \frac{n}{2^n}$ $\left(\frac{\infty}{\infty}\right)$

$$\lim_{n\to\infty} \frac{1}{\underbrace{2^n \cdot \ln 2}_{>0}} = 0$$

(book 76) **ex:** $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n}$

$= \lim_{n\to\infty} (n^{1/n})^2 = \left(\lim_{n\to\infty} (n^{1/n})\right)^2 = 1^2 = 1$

(book 60) **ex:** $\lim_{n\to\infty} \ln n - \ln(n+1)$

$= \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{n+1}\right)$

$= \ln 1 = 0$

(book 65) **ex:** $\lim_{n\to\infty} \frac{n!}{10^{6n}} = \lim_{n\to\infty} \frac{n!}{(10^6)^n} = +\infty$

(book 74) **ex:** $\lim_{n\to\infty} \frac{(10/11)^n}{(9/10)^n + (11/12)^n} = \lim_{n\to\infty} \frac{1}{\left(\frac{9/10}{10/11}\right)^n + \left(\frac{11/12}{10/11}\right)^n}$

$= \lim_{n\to\infty} \frac{1}{\underbrace{\left(\frac{99}{100}\right)^n}_{0} + \underbrace{\left(\frac{121}{120}\right)^n}_{\infty}}$

$= \frac{1}{\infty} = 0$

10.2 INFINITE SERIES

$a_n \rightarrow$ sequence

$a_1 + a_2 + a_3 + \cdots + a_n \rightarrow$ series

$$\left.\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \end{aligned}\right\} \quad a_1 + a_2 + a_3 + \cdots \overset{\text{def}}{=} \lim_{n \to \infty} S_n$$

ex: $a_n = \left(\frac{1}{2}\right)^n \quad , n \geq 1$

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots\right)$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \lim_{n \to \infty} S_n$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n}$$

$$S_1 = 1/2$$

$$S_2 = 1/2 + 1/4 = 3/4$$

$$S_3 = 1/2 + 1/4 + 1/8 = 7/8$$

$$S_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \lim_{n \to \infty} 1 + \frac{1}{2^n} = 1$$

Notation: $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

ex: $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

a_n sequence , $n \geq 1$

$S_n = a_1 + a_2 + \dots + a_n$

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n$$

if limit exists then the series converge otherwise the series diverge.

ex: $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \dots = +\infty$

$$= \lim_{n \to \infty} \frac{n(n+1)}{2}$$

divergent series

Geometric Series

$x \to$ real number

$$x^0 + x^1 + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

$S_n = 1 + x + x^2 + \dots + x^n$

$S_n(1-x) = (1 + x + x^2 + \dots + x^n)(1-x)$

$= 1 + \underline{\underline{x}} + \underline{\underline{x^2}} + \dots + \underline{\underline{x^n}} - (\underline{\underline{x}} + \underline{\underline{x^2}} + \dots + x^{n+1})$

$= 1 - x^{n+1}$

✓ $x \neq 1 \Rightarrow S_n = \dfrac{1 - x^{n+1}}{1-x}$

✓ $x \neq 1 \Rightarrow \sum\limits_{n=0}^{\infty} x^n = \lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} \dfrac{1 - x^{n+1}}{1-x}$

$$\boxed{\sum_{n=0}^{\infty} x^n = \begin{cases} +\infty & , x > 1 \\ \dfrac{1}{1-x} & , -1 < x < 1 \\ \text{diverges} & x \leq -1 \end{cases}}$$

✓ $x = 1 \Rightarrow \sum\limits_{n=0}^{\infty} 1 = 1 + 1 + \dots = +\infty$

ex: $\sum\limits_{n=0}^{\infty} \dfrac{1}{2^n} = \sum\limits_{n=0}^{\infty} \left(\dfrac{1}{2}\right)^n = \dfrac{1}{1 - \frac{1}{2}} = 2$

$= \underbrace{1 + \dfrac{1}{2} + \dfrac{1}{4} + \dots}_{1}$

ex: $\sum\limits_{n=2}^{\infty} \left(-\dfrac{3}{5}\right)^n = \left(\dfrac{-3}{5}\right)^2 + \left(\dfrac{-3}{5}\right)^3 + \dots$

$$= -\left(\frac{-3}{5}\right)^0 - \left(\frac{-3}{5}\right)^1 + \left(\frac{-3}{5}\right)^0 + \left(\frac{-3}{5}\right)^1 + \left(\frac{-3}{5}\right)^2 + \dots$$

$$= -1 + \frac{3}{5} + \sum_{n=0}^{\infty} \left(\frac{-3}{5}\right)^n$$

$$= -1 + \frac{3}{5} + \frac{1}{1 - \left(\frac{-3}{5}\right)} = -1 + \frac{3}{5} + \frac{5}{8}$$

ex: $0.\overline{23} = ?$

$$0.\overline{23} = 0.232323\ldots = 0.23 + 0.0023 + 0.000023 + \ldots$$

$$= \frac{23}{10^2} + \frac{23}{10^4} + \frac{23}{10^6} + \ldots$$

$$= \frac{23}{10^2}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right]$$

$$= \frac{23}{10^2} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{10^2}\right)^n = \frac{23}{100} \cdot \frac{1}{1 - \frac{1}{100}}$$

$$= \frac{23}{100} \cdot \frac{100}{99}$$

$$= \frac{23}{99}$$